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Nonequilibrium Chemical Physics E19a
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Advanced Lab Course
Cooperative Behavior in Networks of Mechanical Oscillators

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1 Introduction

The ability of a system to form spontaneous order is observed in a wide range of systems on different scales. For example, fireflies exhibit an internal natural rhythm which is exposed by them flashing with a certain frequency. That means that each firefly can be considered to be a kind of small oscillator. When a number of fireflies are brought into proximity of one another, one can watch them first flashing in pairs, then groups of three until eventually all fireflies flash in unison [1]. A similar thing has been found in relation to the cardiac rhythm: The human heart has so-called pacemaker cells with an inherent rhythm. The beating of the heart is the result of those cells firing (mostly) simultaneously after experiencing synchronization. (For those interested, a well written introduction to some synchronization phenomena is given in [2].)

A particularly interesting dynamical state found in groups of oscillators is a so-called chimera state: It is characterized by the coexistence of synchrony and disorder. Some of the oscillators are synchronized while the other part of the oscillators exhibit asynchronous behaviour of varying degree. Take for example a *weak* chimera state: Here, the asynchronous group of oscillators shows a phase difference $\Delta\varphi$ which is fluctuating as $t \to \infty$. A 'conventional' chimera state requires the incoherent group to show chaotic behaviour, which can be tricky to verify.

This lab course is aimed to give the students an introduction to some important concepts used in the field of nonlinear dynamics. With a simple set-up using a number of conventional metronomes, the phenomenon of synchronization is illustrated and investigated. The computational approach in the second part of the lab course allows the students to learn about the topic even further.

2 Basics of nonlinear dynamics

The following concepts are mostly taken from [3].
Consider a dynamical system continuous in time. It can be described using a system of differential equations

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, ..., x_n) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, x_2, ...x_n). \end{aligned} \tag{1}$$

Now consider the damped harmonic oscillator. It follows the equation

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0. \tag{2}$$

Using Eq. 1 and the introduction of variables $x_1 = x$ and $x_2 = \dot{x}$, Eq. 2 can be rewritten to

$$\dot{x}_1 = x_2 \tag{3}$$

$$\dot{x}_2 = -\frac{b}{m}x_2 - \frac{k}{m}x_1. \tag{4}$$

Here, the x_i on the right hand side are only present to the first power. The system is considered to be *linear*.
Consider now the swinging of a pendulum following the equation

$$\ddot{x} + \frac{g}{L}\sin(x) = 0. \tag{5}$$

When rewritten like described above, this equation becomes

$$\dot{x}_1 = x_2 \tag{6}$$

$$\dot{x}_2 = -\frac{g}{L}\sin x_1. \tag{7}$$

Here, Eq. 7 features x_2 in a sine function on the right hand side. In general, when the equations include any nonlinear terms of x_i, such as products, powers and other functions, the system is called *nonlinear*.

Fixed points

In a system $\dot{x} = f(x)$ with arbitrary initial conditions, what is the behaviour as $t \to \infty$? To answer this question, one has to solve for so-called *fixed points* that are defined as x^* where $f(x)|_{x^*} = 0$.
There are two kinds of fixed points:

- stable fixed points: all sufficiently small disturbances will approach it
- unstable fixed points: disturbances from these points will grow in time

Bifurcations

The term *bifurcation* refers to qualitative changes in the dynamics of a system as a parameter is varied. Those qualitative changes can be the creation or annihilation of a fixed point or the change in stability of a fixed point.

Example: *The saddle-node bifurcation*
The saddle-node bifurcation is a bifurcation where a pair of fixed points are created/destroyed upon variation of the bifurcation parameter r. The canonical form reads $\dot{x} = r + x^2$. A sketch of the bifurcation is shown below:

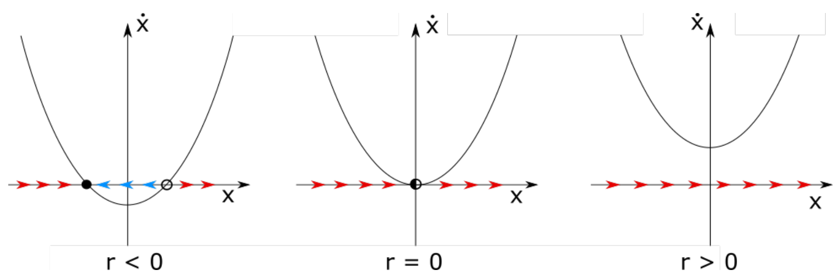


Figure 1: When $r < 0$, the system $\dot{x} = r + x^2$ has two fixed points, one of which is stable and one of which is unstable. At $r = 0$, the stable and unstable fixed points collide. As the resulting fixed point is attracting to $x < 0$ but repelling to $x > 0$, it is called *saddle point.* When r is further increased, the saddle point is annihilated.

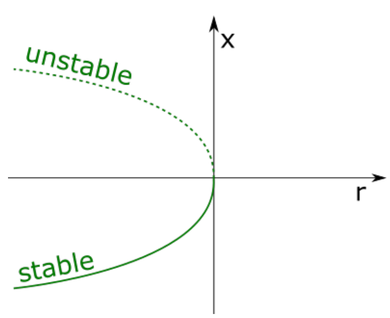


Figure 2: The bifurcation diagram of the saddle-node bifurcation shows the behaviour of the fixed points x^* in relation to the bifurcation parameter r.

In higher dimensions, fixed points can become multidimensional. The system then settles to a stable state which manifests itself as an oscillation. This attracting state is called a *limit cycle.* It is only natural that bifurcations can occur in higher dimensions as well. The so-called *Hopf bifurcation* causes a limit cycle to appear or disappear upon variation of the bifurcation parameter as sketched below.

Kuramoto Model

The following section is based on [4].
Consider a group of N self-driven oscillators where each oscillator has its own natural frequency ω_i. All oscillators are weakly coupled to each other. The long-term behaviour of the oscillator phases θ_i follows the equation

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} \Gamma_{ij}(\theta_j - \theta_i) \tag{8}$$

for $i = 1, ..., N$, where Γ_{ij} is called interaction function. In this model, the interaction function is given by an equal, all-to-all, sinusoidal coupling:

$$\Gamma_{ij}(\theta_j - \theta_i) = \frac{K}{N} \sin(\theta_j - \theta_i). \tag{9}$$

When the coupling strength $K \geq 0$ is relatively low, the oscillator move incoherently and their phases are uniformly distributed. If the coupling strength is increased above a threshold K_c, the system undergoes a phase transition. One part of the oscillators synchronize in a spontaneous manner whereas the remaining oscillators stay incoherent. Full synchronization can be obtained by further increasing the coupling strength. In this state, the oscillators exhibit a mutual frequency despite their different natural frequencies ω_i.

3 Instructions

This lab course is split into two parts: An experimental and a computational part. Due to COVID-19 restrictions, every student is supposed to pick up their own box containing the set-up for the experimental part to take home. This way, the students can conduct the lab course while meeting social distancing requirements.
The students are required to hand in a report after conducting the course. Throughout this instruction manual, exercises and questions that should be brought up in the report are marked as such.

3.1 Metronome set-up

When the students have received the set-up from the instructor, the box should contain the following:

- 4 metronomes
- 4 aluminum barrels
- 2 small planks with bolt and nuts
- 1 large plank
- a container with a number of springs

Notify the instructor in case something is missing.

3.1.1 Oscillators without plank

First, familiarize yourself with the oscillators used in this part of the lab course, i.e. the metronomes. The most important part of the metronome is the weight that is used to adjust the frequency. The handle on the right side of each metronome is used to wind it up. This has to be done regularly throughout the experiment whenever a metronome comes to rest.

Exercise
Wind up two metronomes and adjust them to the same frequency. Nudge them simultaneously while they are located on top of a table or the floor. What do you notice? Did you expect this result? Why or why not?

3.1.2 Oscillators on one plank

Now put one of the two planks on top of two aluminum barrels such that the plank can easily move across. Put two metronomes on top of the plank like so:

and nudge them. You now have a system with two oscillators on a moving surface.

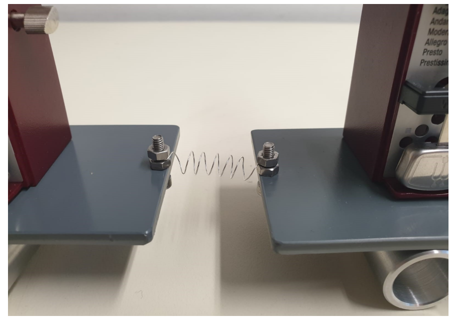
Questions
ⓐ Which parameters does the system have that can change its dynamics? Vary the frequency from low to high. Vary the frequency difference of the metronome pair. Vary the initial phase difference and amplitudes. What do you observe?
ⓑ Now add another metronome and repeat. What is now different compared to before when only 2 metronomes were considered?

Whenever you find a state of interest, videotape it with your smartphone so you can refer to them in your write-up. The videos are to be uploaded together with the report at submission.

3.1.3 Oscillators on two planks coupled with a spring

Another parameter of the system can be introduced by adding another set of metronomes on a plank and connecting the two planks using a spring as pictured above.

Exercise
ⓐ Take a look at the springs that came with the set-up. Find a method to assess the stiffness of the springs in a *qualitative* manner. (You do not need to find the spring constant.)
ⓑ Assemble the two-plank-system coupled with a spring as pictured below and

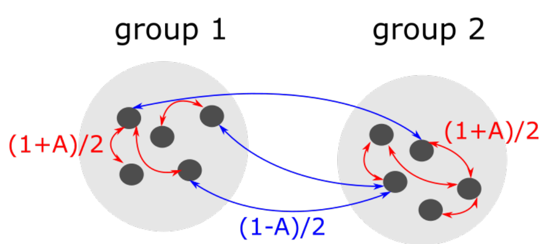


add metronomes to each plank. Repeat the procedure from before and use your previous findings to find interesting states more quickly. Focus especially on the role of spring stiffness. Find the following states and videotape them:

- a fully synchronized state, i.e. $\Delta\varphi = 0$
- an anti-phase state, i.e. two oscillators are in phase respectively but anti-phase to the other two oscillators
- a chimera-type state, i.e. two oscillators are synchronized on one plank while the other two oscillators exhibit disorder

This part of the lab course is based on [5].

3.2 Simulation using Python



In the second part of this lab course you will have a look at a system with two groups of N phase oscillators ($r = const$) with coupling parameters A and β. The oscillators exhibit an *inter*-group coupling (1-A)/2 and an *intra*-group coupling of (1+A)/2. The governing equations for oscillators of the same natural frequency ω are

$$\frac{d\theta_i}{dt} = \omega - \left(\frac{1+A}{2N}\right)\sum_{j=1}^{N}\cos(\theta_i - \theta_j - \beta) - \left(\frac{1-A}{2N}\right)\sum_{j=1}^{N}\cos(\theta_i - \phi_j - \beta), \quad (10)$$

$$\frac{d\phi_i}{dt} = \omega - \left(\frac{1+A}{2N}\right)\sum_{j=1}^{N}\cos(\phi_i - \phi_j - \beta) - \left(\frac{1-A}{2N}\right)\sum_{j=1}^{N}\cos(\phi_i - \theta_j - \beta) \quad (11)$$

where θ_i are the phases of the oscillators in group 1, ϕ_i the phases of group 2.

Exercise

ⓐ Download the code (the link will be provided by the instructor via mail). The code is written for N of 2, 3 and 4. It simulates the system and plots the oscillator phase time series.

ⓑ Run the code several times for one parameter setting. What do you notice and why?

ⓒ Play around with the parameters N, A and β. How do the dynamics change? (Hint: What is a bifurcation?)

Question

Consider the complex number $z = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j}$ and its absolute value $|z|$. Regarding a dynamical system, what does $|z|$ represent?

In the code, you can compute $z(t)$ by calling the function `z(phases)` with the time series as an argument.

Exercise

ⓐ Leave N fixed and run the simulation for different values of A and β. Look at both the time series and $|z(t)|$. Interpret and summarize your findings qualitatively in a A-β-diagram.

ⓑ Take a look at what happens at N= 3 and $\beta = 0.1$ when going stepwise from $A = 0.3$ to $A = 0.2$. (Hint: What is a Hopf-bifurcation?)

ⓒ Extend the code for arbitrary N > 4 using Eq. 10 and 11. Compute $|z(t)|$ for some different N and interpret the result. What happens for large N?

References

[1] John Buck and Elisabeth Buck. *Biology of synchronous flashing of fireflies.* 211:562-564, 1966.

[2] Steven H Strogatz. *SYNC: How order emerges from chaos in the universe, nature, and daily life.* 1st ed. Hyperion, New York 2003.

[3] Steven H Strogatz. *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering.* 2nd ed. Westview Press, Boulder 2015.

[4] Yoshiki Kuramoto. *Chemical Oscillations, Waves, and Turbulence.* Springer, Berlin 1984.

[5] Erik A Martens, Shashi Thutupalli, Antoine Fourrière, and Oskar Hallatschek. *Chimera states in mechanical oscillator networks.* PNAS 110 (26) 10563-10567, 2013.