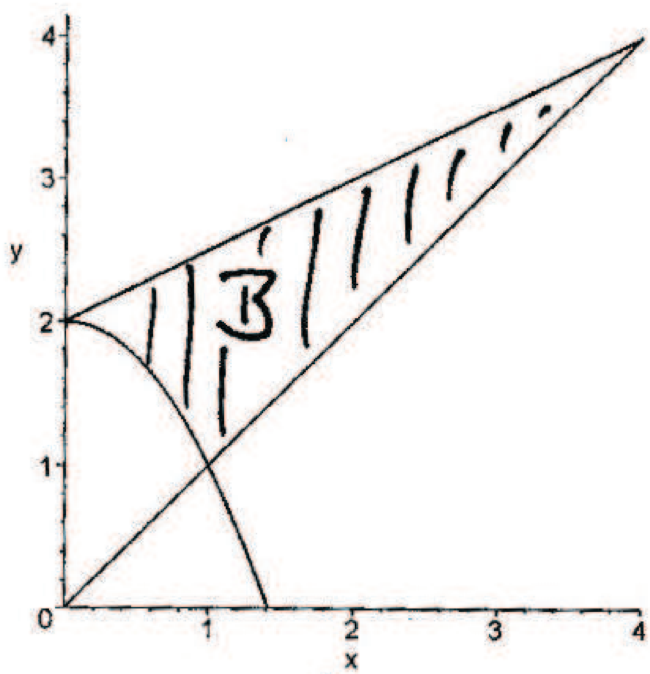
Integralrechung

Aufgabe 1

$$
\begin{aligned}
\int_D f\,dx &= \int_0^{\frac{1}{3}} \left(\int_0^{\frac{-1}{2}} \Big(\int_0^1 (x_1 + 2x_2 + 3x_3)^2 dx_1 \Big) dx_2 \right) dx_3 \\
&= \int_0^{\frac{1}{3}} \left(\int_0^{\frac{-1}{2}} \frac{1}{3} \Big((1 + 2x_2 + 3x_3)^3 - (2x_2 + 3x_3)^3 \Big) dx_2 \right) dx_3 \\
&= \int_0^{\frac{1}{3}} \frac{1}{24} \left((1 + 3x_3)^4 - 2(3x_3)^4 + (3x_3 - 1)^4 \right) dx_3 \\
&= \frac{1}{24.15} (2^5 - 2 - 1 + 1) = \frac{1}{12}
\end{aligned}
$$

Aufgabe 2



$$\int\int_B \left(1-\frac{2x}{y}\right)\ dF = \int_{y=1}^{y=2}\left[\int_{x=\sqrt{2-y}}^{x=y}\left(1-\frac{2x}{y}\right)dx\right]dy + \int_{y=2}^{y=4}\left[\int_{x=2(y-2)}^{x=y}\left(1-\frac{2x}{y}\right)dx\right]dy = \int_{y=1}^{y=2}\left(\frac{2}{y}-1-\sqrt{2-y}\right)dy + \int_{y=2}^{y=4}\left(\frac{16}{y}-12+2y\right)dy = -\frac{41}{3}+18\ln 2$$

Aufgabe 3

$$\int_C f(x)\ ds = \int_{t_a}^{t_e} f(x(t))\|\dot{x}(t)\|_2\ dt$$

$$\|\dot{x}(t)\|_2 = \sqrt{(-\sin t)^2+(\cos t)^2+(\sinh t)^2} = \sqrt{1+\sinh^2 t} = \cosh t$$

$$f(x(t) = \cos^2 t + \sin^2 t + \frac{1}{\cosh t} = 1 + \frac{1}{\cosh t}$$

$$\int_C f(x)\ ds = \int_0^{2\pi}\left(1+\frac{1}{\cosh t}\right)\cosh t\ dt = \sinh t + t\Big|_0^{2\pi} = \sinh 2\pi + 2\pi$$

Aufgabe 4

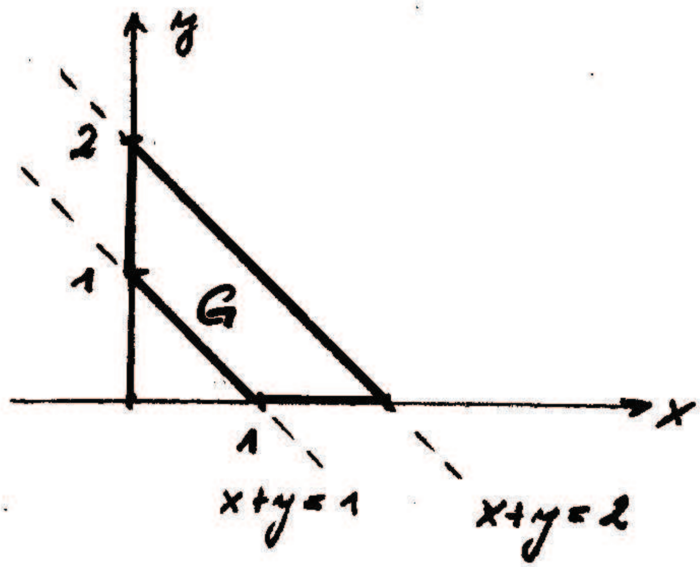
2.a)-

$$L = \int_0^{4\pi} |\dot{\mathbf{S}(t)}|dt = \int_0^{4\pi}\left|\begin{pmatrix}-\sin t\\ \cos t\\ 2\end{pmatrix}\right|dt = \int_0^{4\pi}\sqrt{1+4}dt = 4\sqrt{5}\pi$$

2.b)-

$$\int_{\mathbf{S}} \mathbf{V}(x).d\mathbf{x} = \int_0^{4\pi}\begin{pmatrix}\sin t\cos t\\ 2t\sin t\\ 2t\cos t\end{pmatrix}\mathbf{.}\begin{pmatrix}-\sin t\\ \cos t\\ 2\end{pmatrix}dt = \int_0^{4\pi}(-\sin^2 t\cos t + 2t\sin t\cos t + 4t\cos t)\ dt = \left[-\frac{1}{3}\sin^3 t.t\cos^2 t + \frac{1}{2}\cos t\sin t + \frac{1}{2}t + 4\cos t + 4t\sin t\right]_0^{4\pi} = -4\pi + 2\pi + 4 - 4 = -2\pi$$

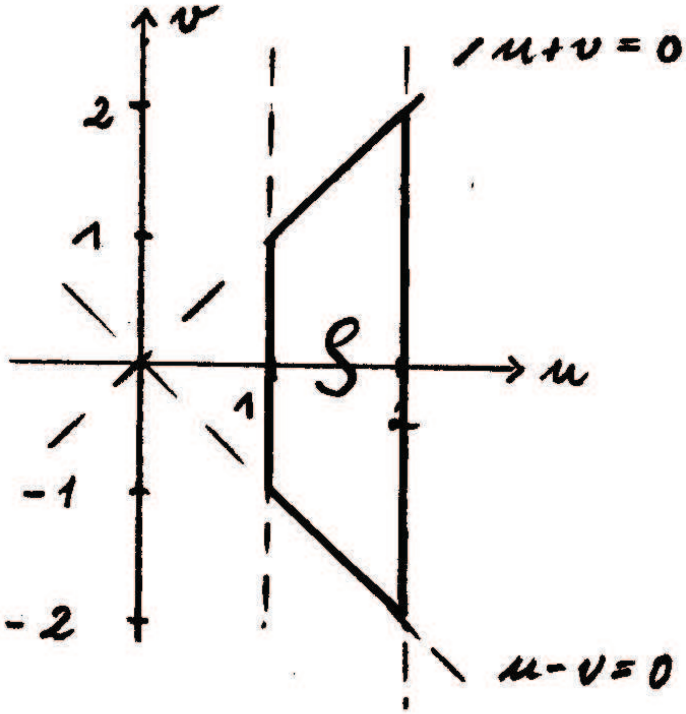
Aufgabe 5



a)

b)

$$x = \frac{1}{2}(u+v)$$
$$y = \frac{1}{2}(u-v)$$
$$(1)+(1) \Rightarrow u = x+y$$
$$(1)-(1) \Rightarrow u = x-y$$



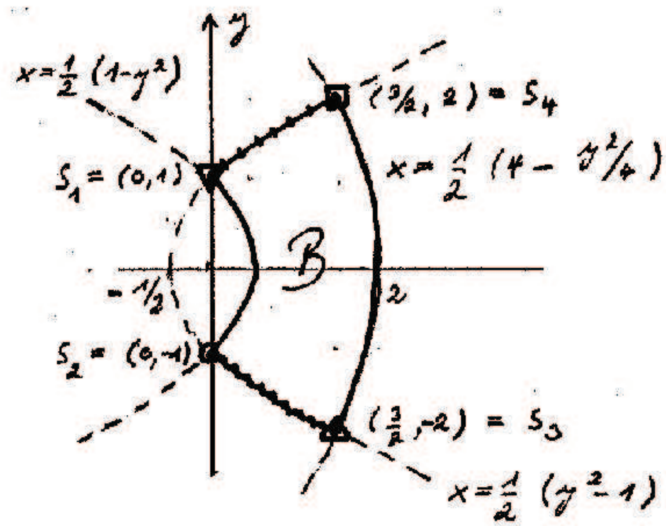
c)

$$x_u y_v - x_v y_u = -\frac{1}{2}.\frac{1}{2} - \frac{1}{2}.\frac{1}{2} = -\frac{1}{2}$$

d)

$$\begin{aligned}
\int_G \exp\left(\frac{x-y}{x+y}\right) dxdy &= \int\int_S \exp\left(\frac{v}{u}\right).|x_u y_v - x_v y_u| dudv \\
&= \frac{1}{2}\int_{u=1}^{u=2}\int_{v=-u}^{v=u} \exp\left(\frac{v}{u}\right) dvdu \\
&= \frac{1}{2}\int_{u=1}^{u=2} u\exp\left(\frac{v}{u}\right)|_{v=-u}^{u} du \\
&= \sinh 1 \int_{u=1}^{u=2} udu = \frac{3}{2}\sinh 1
\end{aligned}$$

Aufgabe 6



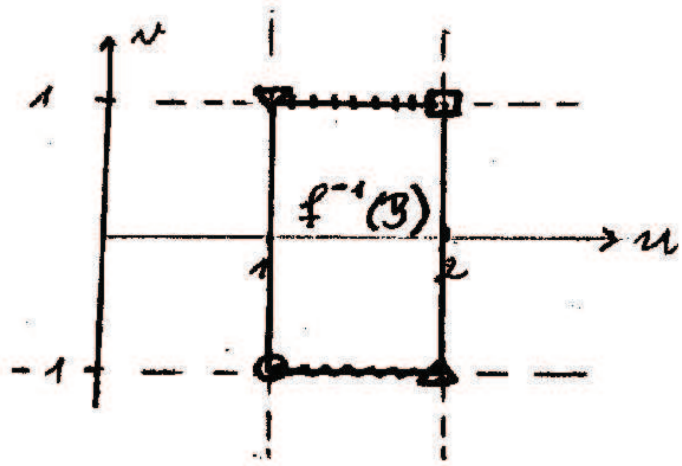
Die Parabolen sind von der Form

$$\begin{aligned} x &= \frac{1}{2}(u^2 - v^2) \\ y &= \frac{1}{2}(uv) \end{aligned}$$

- $$\begin{aligned} x &= \frac{1}{2}(y^2 - 1) = \frac{1}{2}((uv)^2 - 1) \doteq \frac{1}{2}(u^2 - v^2) \\ \Rightarrow v^2 + u^2v^2 - u^2 - 1 &= 0 \\ \Rightarrow (u^2+1)(v^2-1) &= 0 \\ \Rightarrow v^2 &= 1 \end{aligned}$$

- $$\begin{aligned} x &= \frac{1}{2}(1 - y^2) = \frac{1}{2}(1 - (uv)^2) \doteq \frac{1}{2}(u^2 - v^2) \\ \Rightarrow 1 - u^2v^2 &= u^2 - v^2 \qquad \text{mit} \quad v^2 = 1 \\ \Rightarrow u &= 1 \end{aligned}$$

- $$\begin{aligned} x &= \frac{1}{2}\left(4 - \frac{y^2}{4}\right) \doteq \frac{1}{2}(u^2 - v^2) \\ \Rightarrow 16 - u^2 &= u^2 \\ \Rightarrow u &= 2 \end{aligned}$$



$$Df(u,v) = \begin{pmatrix} u & -v \\ v & u \end{pmatrix}$$
$$detDf(u,v) = u^2 + v^2 > 0 \Rightarrow \text{ f bijektiv}$$

$$\begin{aligned} F &= \int_B dxdy = \int_{f^{-1}(B)} |detDf(u,v)| \; dudv \\ &= \int_{u=1}^{2} \int_{v=-1}^{1} (u^2+v^2) \; dvdu = \frac{16}{3} \end{aligned}$$

Aufgabe 7

Die Funktionalmatrix ist :

$$\begin{aligned} Df(r,\varphi,\theta) &= \begin{pmatrix} a\cos\varphi\cos\theta & -ar\sin\varphi\cos\theta & -ar\cos\varphi\sin\theta \\ b\sin\varphi\cos\theta & br\cos\varphi\cos\theta & -br\sin\varphi\sin\theta \\ c\sin\theta & 0 & cr\cos\theta \end{pmatrix} \\ \text{mit} \quad detDf(r,\varphi,\theta) &= abcr^2\cos\theta \end{aligned}$$

Die Volumen ist :

$$\begin{aligned} V(E) &= \int_E dxdydz = \int_{r=0}^{1} \int_{\varphi 0}^{2\pi} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} abcr^2\cos\theta \; d\theta d\varphi dr \\ &= abc\frac{r^3}{3}\bigg|_0^1 .2\pi.\sin\theta\bigg|_{\frac{\pi}{2}}^{-\frac{\pi}{2}} = \frac{4\pi}{3}abc \end{aligned}$$

Der axialen Trägheitsmoment bezüglich der z-Achse : $T_z = \int_E \varrho \ dxdydz$ mit $\varrho = \varrho(x, y, z)$ Abstand der Punkte (x, y, z) von der z-Achse

$$\varrho^2 = x^2 + y^2 = a^2 r^2 \cos^2\varphi \cos^2\theta + b^2 r^2 \sin^2\varphi \cos^2\theta$$

Damit ist:

$$\begin{aligned}
T_z &= \int_{r=0}^{1} \int_{\varphi 0}^{2\pi} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} (a^2\cos^2\varphi + b^2\sin^2\varphi) r^2 \cos^2\theta abc r^2 \cos\theta \ d\theta d\varphi dr \\
&= abc \int_{r=0}^{1} r^4 \ dr \int_{\varphi 0}^{2\pi} (a^2\cos^2\varphi + b^2\sin^2\varphi) \ d\varphi \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\theta \ d\theta \\
&= abc.\frac{1}{5}.\left(\frac{a^2-b^2}{4}\sin 2\varphi + \frac{a^2+b^2}{2}\varphi\right)\Bigg|_{\varphi 0}^{2\pi}.\left(\sin\theta - \frac{1}{3}\sin^3\theta\right)\Bigg|_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{4}{15}\pi abc(a^2+b^2)
\end{aligned}$$

Aufgabe 8

a) Wir betrachten Zylinderkoordinaten $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix}$ Die Gesamtmasse von Z ist

$$\begin{aligned}
M &= \int_Z \varrho dx = \int_0^1 \int_0^{2\pi} \int_0^2 (2 - x_3) dx \\
&= \int_Z \varrho dx = \int_0^1 \int_0^{2\pi} \int_0^2 (2 - z) \ r dr d\varphi dz \\
&= \int_0^1 \int_0^{2\pi} (2 - z) z d\varphi dz = 4\pi \int_0^1 (2 - z) \ dz \\
&= 4\pi.\frac{3}{2} = 6\pi
\end{aligned}$$

b) Wir verwenden wieder Zylinderkoordinaten. Nach Definition sind die Koordinaten des Schwerpunktes gegeben durch

$$S_i = \frac{1}{M}\int_Z x_i \varrho dx \qquad i = 1, 2, 3$$

wobei M die Masse des gesamten Zylinders ist

Also sind :

$$S_1 = \frac{1}{M}\int_0^1\int_0^2\int_0^{2\pi} r\cos\varphi(2-z)\ r\ \ d\varphi dr dz = 0$$

$$S_2 = \frac{1}{M}\int_0^1\int_0^2\int_0^{2\pi} r\sin\varphi(2-z)\ r\ \ d\varphi dr dz = 0$$

$$\begin{aligned} S_3 &= \frac{1}{M}\int_0^1\int_0^2\int_0^{2\pi} z(2-z)\ r\ \ d\varphi dr dz \\ &= \frac{4\pi}{M}\int_0^1 z(2-z)dz = \frac{4\pi}{M}.\frac{2}{3} = \frac{4\pi}{6\pi}.\frac{2}{3} = \frac{4}{9} \end{aligned}$$

Aufgabe 9

Wir führen Kugelkoordinaten (r, θ, φ) ein, so dass $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} r\sin\theta\cos\varphi \\ r\sin\theta\sin\varphi \\ r\cos\theta \end{pmatrix}$
Durch einsetzen die neuen Koordinaten in der Gleichung der Sphäre bekommen wir:

$$r^2 = 16$$

Die Gleichung von dem Kegel lautet dann:

$$\begin{aligned} r\cos\theta &= \sqrt{r^2\sin^2\theta} \\ \Rightarrow r\cos\theta &= r\sin\theta \\ \Rightarrow \tan\theta &= 1 \\ \Rightarrow \theta &= \frac{\pi}{4} \qquad \text{(Öffnungswinkel des Kegels)} \end{aligned}$$

Das Volumen von K ist dann :

$$\begin{aligned} V &= \int_K dx = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^4 r^2\sin\theta\ \ dr d\theta d\varphi \\ &= \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{64}{3}\sin\theta\ \ d\theta d\varphi = \frac{64}{3}\int_0^{2\pi}\Big(1-\frac{\sqrt{2}}{2}\Big)d\varphi \\ &= \frac{64}{3}.2\pi.\Big(1-\frac{\sqrt{2}}{2}\Big) = \frac{64}{3}\pi(2-\sqrt{2}) \end{aligned}$$